EXERCISE SHEET – 1

Due on 4th September 2026, by 5 pm IST.

(1) What is the maximum number of -1's that an ASM can have? Construct such a matrix attaining this maximum value. How many ASMs of order n can attain this maximum value?

(2) Show that the formula for A_n follows from the formula for $A_{n,k}$.

(3) Show that there are only 20 distinct possibilities for the orientations of the σ_i's in the Yang-Baxter equation.

(4) Show that there are the following five equivalence classes for the possibilities of the orientations of the σ_i's in the Yang-Baxter equation:

(a) $\{1,2,3\}, \{4,5,6\}, \{2,1,6\}, \{3,4,5\}$,

(b) $\{2,3,4\}, \{5,6,1\}$,

(c) $\{1,3,5\}, \{2,4,6\}$,

(d) $\{1,2,4\}, \{4,5,1\}, \{2,1,5\}, \{3,5,6\}, \{2,3,6\}, \{1,3,6\}, \{3,4,6\}, \{2,4,5\}$,

(e) $\{2,3,5\}, \{5,6,2\}, \{1,6,4\}, \{1,3,4\}$.

(5) Prove the Yang-Baxter equation for the classes (d) and (e) above.

(6) A **monotone triangle** of order n is an equilateral triangular array of numbers with bottom row $1, 2, \ldots, n$ that satisfies the inequality $\leq$ along all southwest to northeast diagonals, the inequality $\leq$ along all northwest top southeast diagonals, and the inequality $<$ across all rows from left to right.

Find a bijection between ASMs of order n and monotone triangles of order n.

(7) A **fully packed loop** (FPL) is a graph on an $n \times n$ grid of vertices with $2n$ exterior edges at every other spot such that each interior vertex has degree 2. Can you show the correspondence between ASMs and FPLs?